Analysis 1

29 November 2023

Warm-up: Give the derivative of

$$f(x) = 8\sin(x) + 6\sqrt{x}.$$

$$f' = 8(\sin(x))' + 6(x^{1/2})'$$

$$= 8\cos(x) + 6\left(\tfrac{1}{2}x^{-1/2}\right)$$

$$= 8\cos(x) + \frac{3}{\sqrt{x}}$$

There are several ways to combine functions:

- SUM: $\sin(x) + \sqrt{x}$.
- DIFFERENCE: $\sin(x) - \sqrt{x}$, and $\sqrt{x} - \sin(x)$.
- PRODUCT: $\sqrt{x} \cdot \sin(x)$.
- QUOTIENT: $\dfrac{\sin(x)}{\sqrt{x}}$, and $\dfrac{\sqrt{x}}{\sin(x)}$.
- COMPOSITION: $\sin\left(\sqrt{x}\right)$, and $\sqrt{\sin(x)}$.

We know how to find derivatives of these already.

TODAY

Later

Students on the left:

1. Find $(x^3)'$, meaning the derivatives of x^3.
2. Find $(x^2)'$, meaning the derivatives of x^2.
3. Simplify $(x^3)' \cdot (x^2)'$.

Students on the right:

1. Simplify $x^3 \cdot x^2$.
2. Find the derivative of the function from your step 1.

$(f \cdot g)'$ is NOT $f' \cdot g'$.

Everyone:

1. Find $(x^3)'$.

2. Find $(x^2)'$.

3. Simplify $(x^3) \cdot (x^2)' + (x^3)' \cdot (x^2)$.

Not derivatives!

Simplify $(x^3)(2x) + (3x^2)(x^2)$

Product Rule

$$(f \cdot g)' = f \cdot g' + f' \cdot g$$

We can write the Product Rule with prime notation or fraction notation:

Product Rule

$$(f \cdot g)' = f \cdot g' + f' \cdot g$$

Product Rule

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[fg\right] = f\frac{\mathrm{d}g}{\mathrm{d}x} + \frac{\mathrm{d}f}{\mathrm{d}x}g$$

We have just checked that this rule is true for $f(x) = x^2$ and $g(x) = x^3$.

Example: What is the derivative of $x^8 \sin(x)$?

$fg' + f'g = (x^8)(\cos(x)) + (8x^7)(\sin(x))$.

We could also write this as $x^7(x\cos(x) + 8\sin(x))$.

You do ***not*** need to understand why the Product Rule is true (you only need to be able to use it). But if you are curious…

Algebra proof

$$\lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

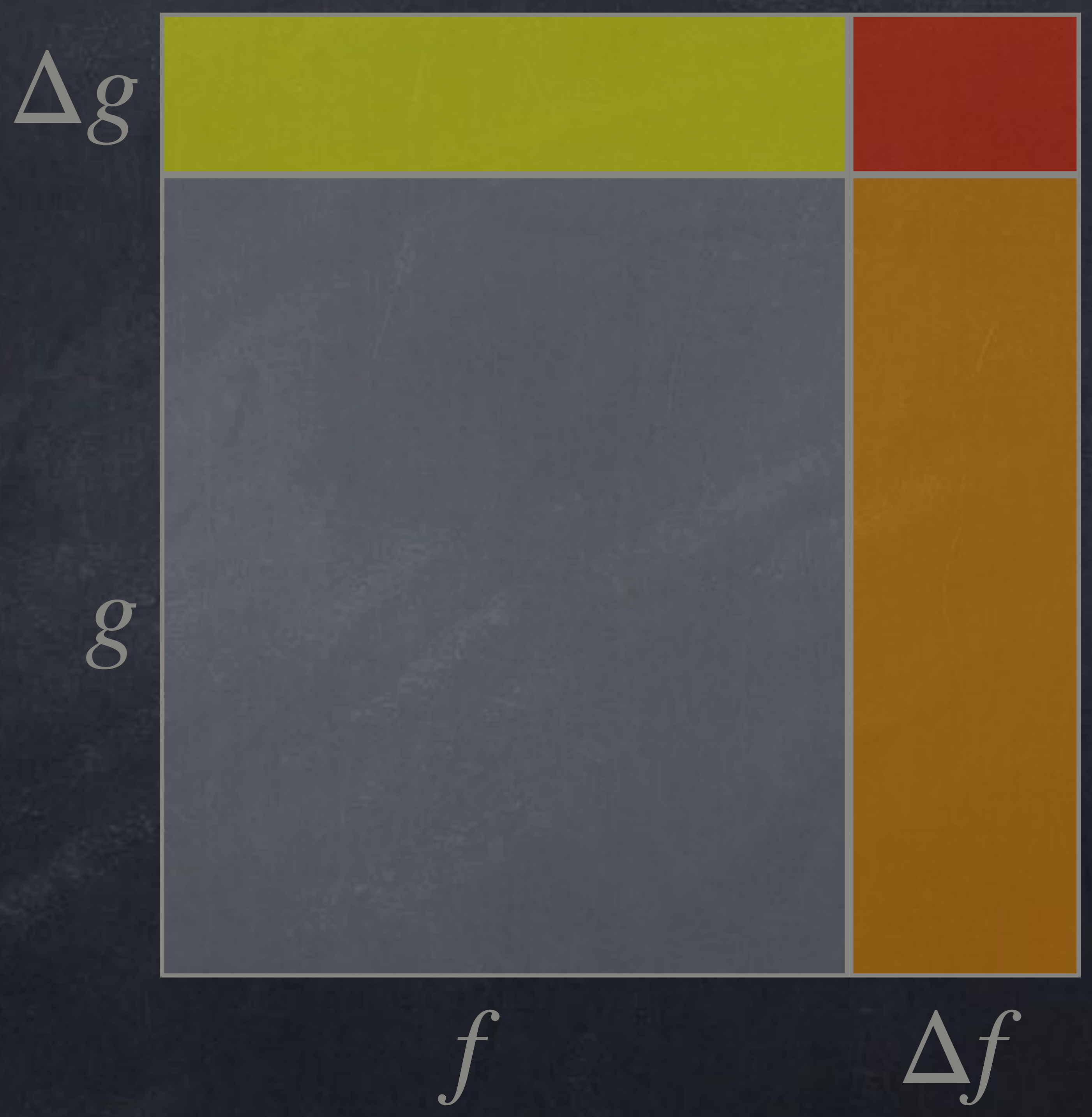
$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{\big(f(x+h) - f(x)\big)g(x+h) + f(x)\big(g(x+h) - g(x)\big)}{h}$$

$$= \left(\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}\right) g(x) + f(x) \left(\lim_{h\to 0} \frac{\big(g(x+h) - g(x)\big)}{h}\right)$$

$$= f'(x)g(x) + f(x)g'(x)$$

Visual proof



$$\Delta(f \cdot g) \approx f \cdot \Delta g + \Delta f \cdot g$$

Which of these are products of numbers?

- $5 \cdot 7$
- $5 \cdot (1 + 2)$
- $(3 \cdot 2) + 7$

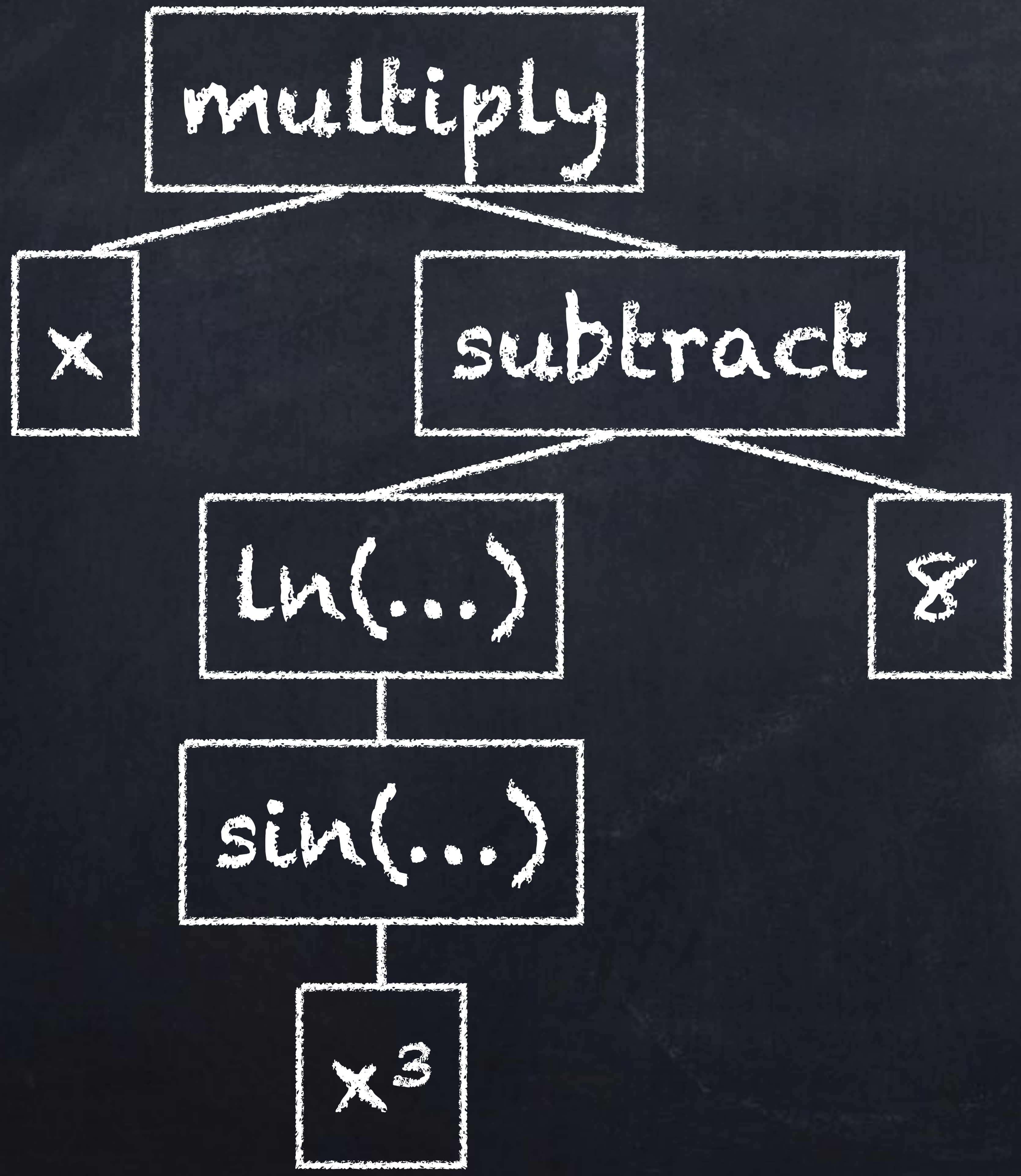
Which of these are products of functions?

- $e^x \sin(x)$ Yes
- $\sqrt{x^7} + x^3$ No*
- $x^3 \ln(x)$
- $x^3\left(x^2 - \cos(x^3)\right)$
- $(3x - 7)(2x + 1)^5$
- $x \ln(\sin(x^3 - 8))$
- $x \ln(\sin(x^3) - 8)$
- $x \ln(\sin(x^3)) - 8$

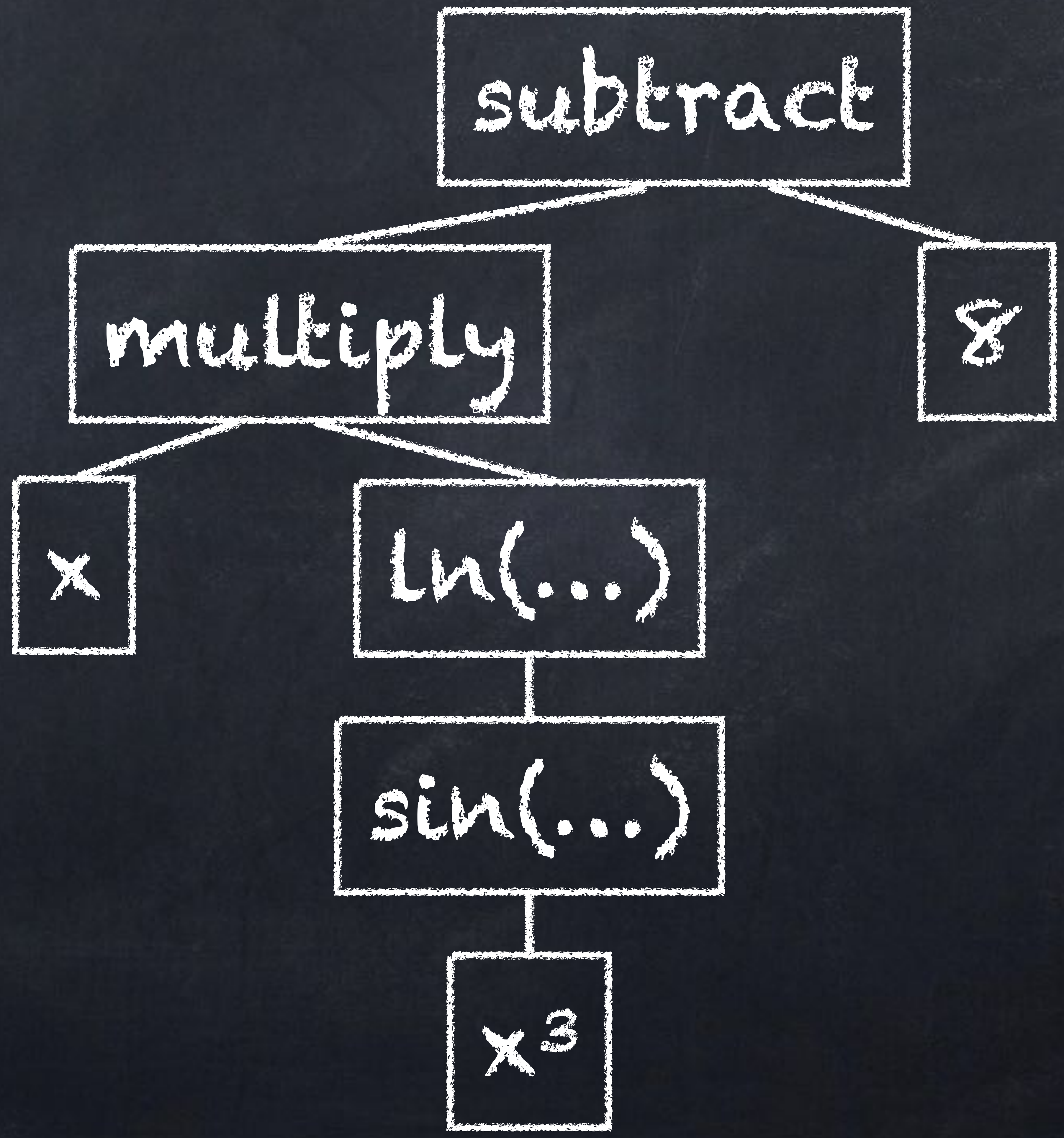
* Technically anything can be a product because you can multiply by 1. The point is that we would not have to use the Product Rule for any of the “No” expressions here.

It might help to think of an expression as a "tree":

$$x \ln(\sin(x^3) - 8)$$



$$x \ln(\sin(x^3)) - 8$$



Higher derivatives

The **second derivative** of a function is the derivative of its derivative.

We can write this as

- f'' because it is $(f')'$, or
- $\dfrac{\mathrm{d}^2 f}{\mathrm{d}x^2}$ because it is $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\dfrac{\mathrm{d}}{\mathrm{d}x}[f]\right]$.

Speaking:
"f double-prime"

To calculate a second derivative, just differentiate *twice*!

Example: for $f(x) = 9x^4$ we have $f''(x) = 108x^2$ because

$$9x^4 \quad \rightsquigarrow \quad 36x^3 \quad \rightsquigarrow \quad 108x^2.$$

Higher derivatives

The **third derivative** of a function is the derivative of its second derivative.

We can write this as

- f''' because it is $(f'')'$, or
- $\dfrac{\mathrm{d}^3 f}{\mathrm{d}x^3}$ because it is $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\dfrac{\mathrm{d}^2 f}{\mathrm{d}x^2}\right]$.

Speaking:
"f triple-prime"

To calculate a second derivative, just differentiate *three times*!

Example: for $f(x) = 9x^4$ we have $f'''(x) = 216x$ because

$$9x^4 \rightsquigarrow 36x^3 \rightsquigarrow 108x^2 \rightsquigarrow 216x.$$

Higher derivatives

We know $f'(x)$ can tell us whether $f(x)$ is increasing or decreasing.

What can $f''(x)$ tell us?

What can $f'''(x)$ tell us?

If $f(t)$ is position then
$f'(t)$ is velocity or speed,
$f''(t)$ is acceleration,
$f'''(t)$ is jerk,
$f^{(4)}(t)$ is snap or jounce,
$f^{(5)}(t)$ is crackle,
$f^{(6)}(t)$ is pop.

Finding local extremes

Last time

To find the local min/max of $f(x)$,

1. Find the critical points of f.

2. Compute signs of f' somewhere in between each CP, and at one point with $x <$ all CP, and at one point with $x >$ all CP.

3. **The First Derivative Test**

- If $f' > 0$ to the left of $x = c$ and $f' < 0$ to the right of $x = c$, then f has a local maximum at $x = c$.
- If $f' < 0$ to the left of $x = c$ and $f' > 0$ to the right of $x = c$, then f has a local minimum at $x = c$.
- If f' has the same sign on both sides of $x = c$, then f has *neither* a local minimum nor local maximum at $x = c$.

Last time

Task 1: Given that the critical points of

$$g(x) = \frac{1}{5}x^5 - 2x^3 + 4x^2 - 3x$$

are -3 and 1, classify each as a local minimum, local maximum, or neither.

$$g' = x^4 - 6x^2 + 8x - 3$$

x		−3		1	
f	↗	local max	↘	local min	↗
f′	+	0	−	0	+

New task: Given that $x = \frac{3}{2}$ is a critical point of

$$f(x) = x^6 - \frac{9}{5}x^5 - \frac{15}{2}x^4 + 15x^3 + 18x^2 - 54x + 5,$$

classify it as a local minimum, local maximum, or neither.

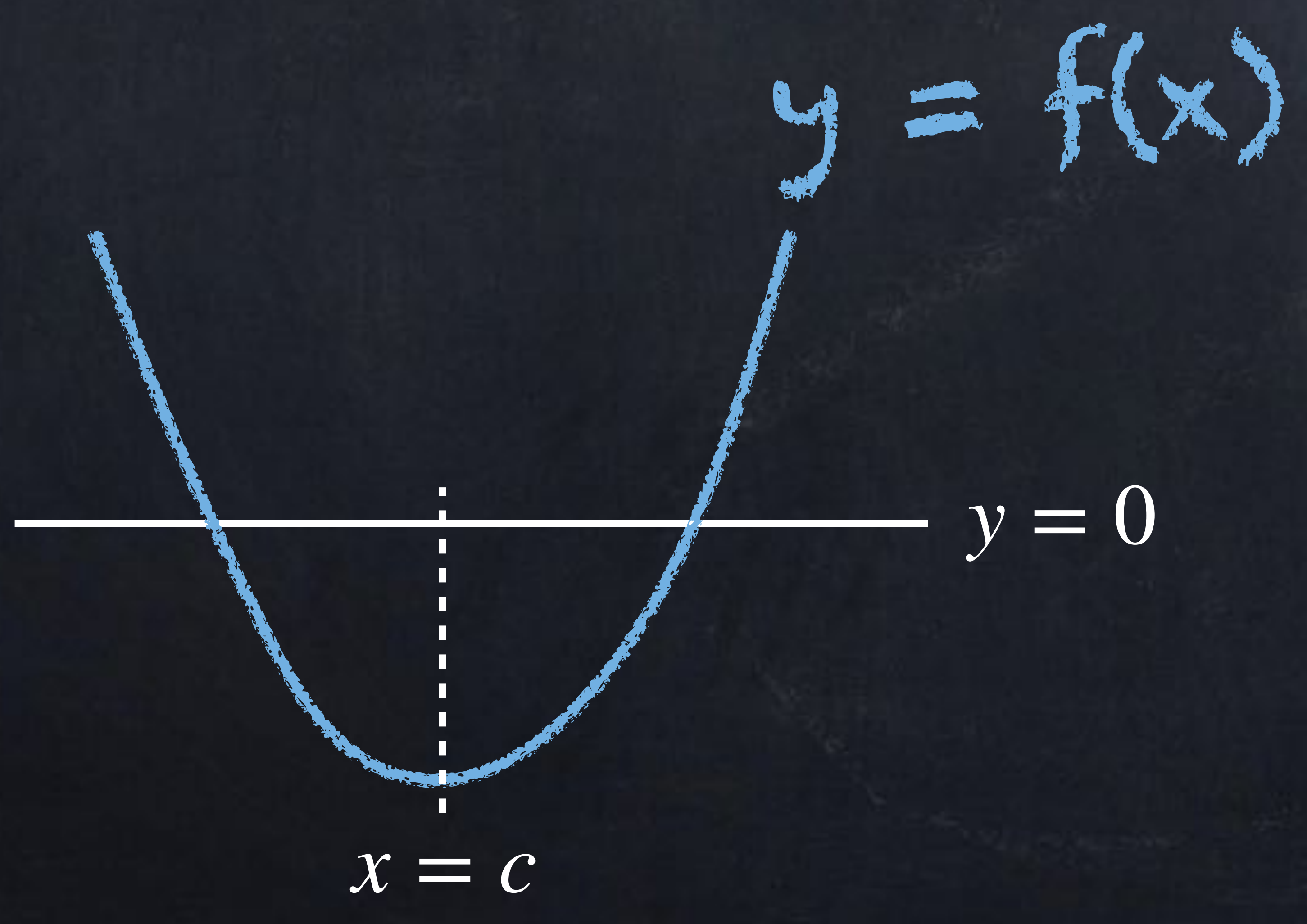
$$f' = 6x^5 - 9x^4 - 30x^3 + 45x^2 + 36x - 54$$

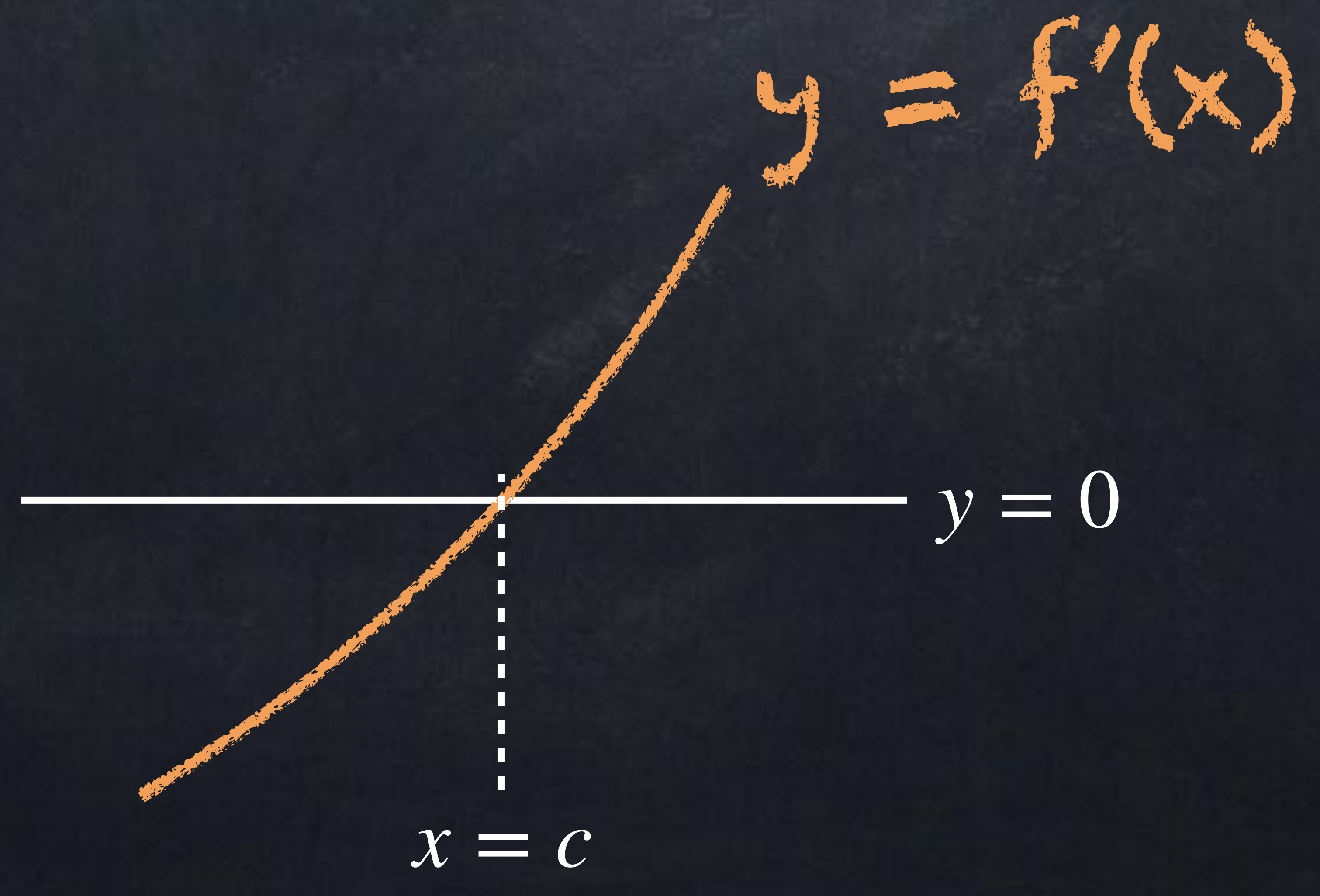
x		1.5	
f			
f'		0	

Finding local extremes

There are some problems with the First Derivative Test.

- Practical: we need to know all the CP of f in some interval in order to be sure we do not "skip over" any when calculating f' at points.
- Philosophical: we are looking for a *local* property, so why are we doing anything at points with $x \neq c$ to classify what kind of CP $x = c$ is?





If $g(x)$ is negative for $x < 6$ and $g(x)$ is positive for $x > 6$ then…

- Could $g'(6)$ be positive?
- Could $g'(6)$ be negative?
- Could $g'(6)$ be zero?

Finding local extremes

To find the local min/max of $f(x)$,

1. Find the CPs of f.

2. Compute signs of f' somewhere in each interval.
3. **The First Derivative Test**

or

2. Compute signs of f'' at each CP.
3. **The Second Derivative Test**

Finding local extremes

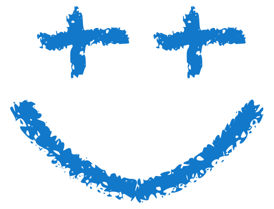
To find the local min/max of $f(x)$,

1. Find the CPs of f.

2. Compute (the signs of) the values of f'' at each CP.

3. **The Second Derivative Test**

- If $f'(c) = 0$ and $f''(c) < 0$ then f has a local maximum at $x = c$. 

- If $f'(c) = 0$ and $f''(c) > 0$ then f has a local minimum at $x = c$. 

(The test does not help if $f''(c) = 0$.)

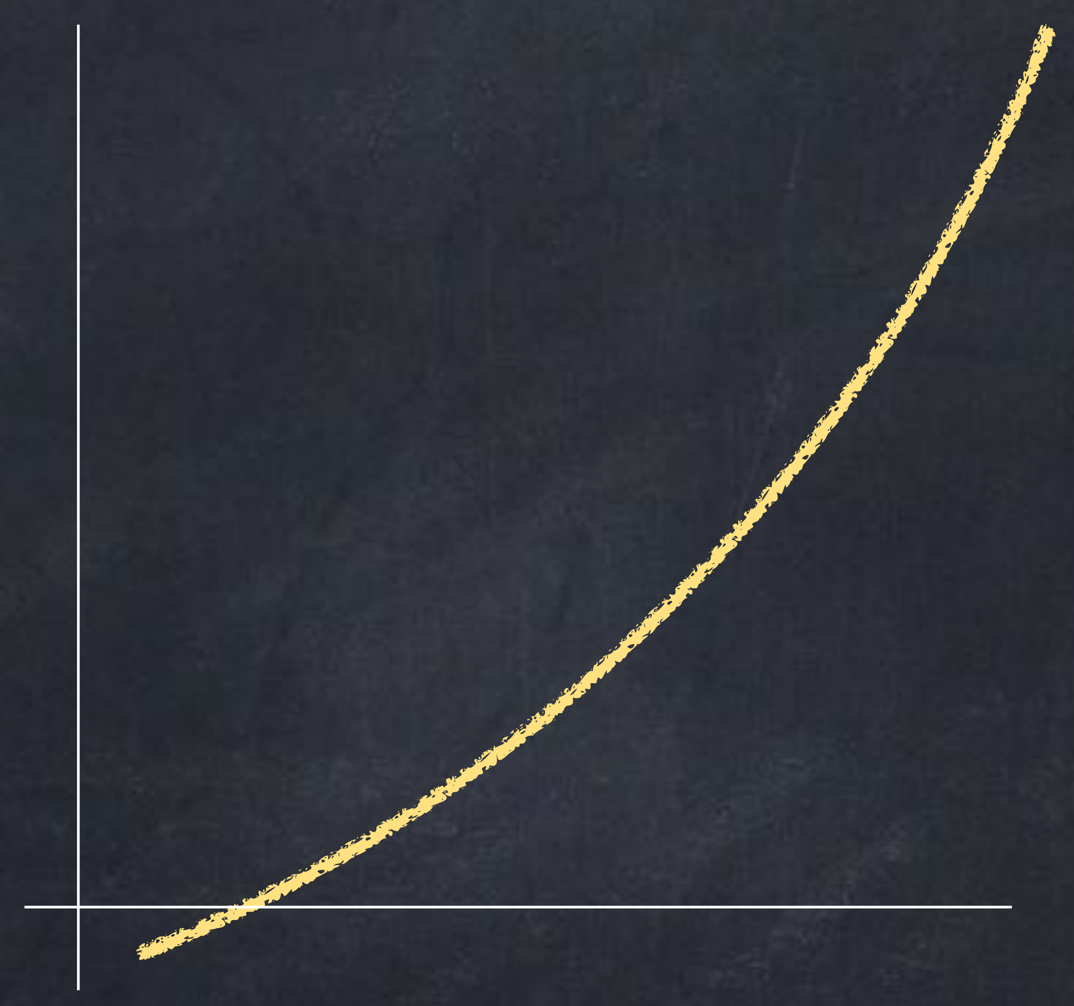
Task: Find the critical points of

$$f(x) = 2x^3 - \frac{3}{2}x^2 - 135x + 22$$

and classify each one as a local minimum or local maximum.

Concavity

These two functions are both increasing:



They both have $f'(x) > 0$, but clearly their behavior is different.

Concavity

There are several official definition for “concave up” and “concave down”.
We will just use pictures.

concave up

also called “convex”
or “convex down”

down

concave down

also called “concave”
or “convex up”

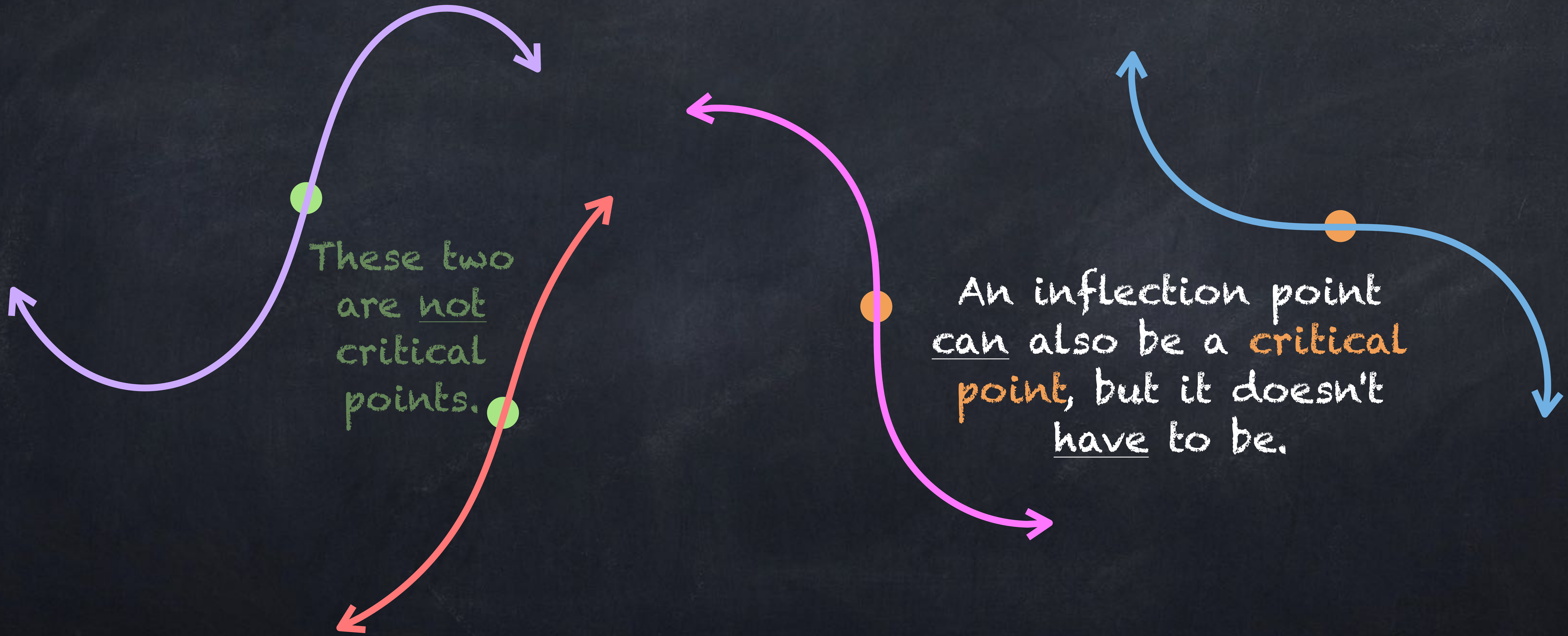
If $f''(x) > 0$ then f is concave up.

If $f''(x) < 0$ then f is concave down.

Image source: https://www.mathsisfun.com

Concavity

Definition: an **inflection point** is a point where the concavity changes.



Derivative

Give the derivatives of the following functions:

- x^2
- $5x^4$
- $2x^\pi$
- 2^x
- $\sqrt{x}$

Find the derivative of 2^x using $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

What about $\frac{\mathrm{d}}{\mathrm{d}x}\left[3^x\right]$?

The Exponential Rule

If a is a constant, the derivative of a^x is $a^x \ln(a)$.

In particular, $(e^x)' = e^x$.